

INVERSE CLOSED ULTRADIFFERENTIAL SUBALGEBRAS

ANDREAS KLOTZ

ABSTRACT. In previous work we have shown that classical approximation theory provides methods for the systematic construction of inverse-closed smooth subalgebras. Now we extend this work to treat inverse-closed subalgebras of ultradifferentiable elements. In particular, Carleman classes and Dales-Davie algebras are treated. As an application the result of Demko, Smith and Moss and Jaffard on the inverse of a matrix with exponential decay is obtained within the framework of a general theory of smoothness.

1. INTRODUCTION

We describe new methods to generate a smooth inverse-closed subalgebra of a given Banach algebra \mathcal{A} and to characterize this subalgebra by approximation properties and by weights. Recall that a subalgebra \mathcal{B} of \mathcal{A} is inverse-closed in \mathcal{A} , if

every $b \in \mathcal{B}$ that is invertible in \mathcal{A} is actually invertible in \mathcal{B} .

A prototype of an inverse-closed subalgebra is the Wiener algebra of absolutely convergent Fourier series, which is inverse-closed in the algebra of continuous functions on the torus. Another example is the algebra $C^1(\mathbb{T})$ of continuously differentiable functions on the torus; the proof that $C^1(\mathbb{T})$ is inverse-closed in $C(\mathbb{T})$ is essentially the quotient rule of classical analysis.

Many methods for the construction of inverse-closed subalgebras are based on generalizations of this simple smoothness principle. In the context of Banach algebras, derivatives are replaced by derivations. The Leibniz rule for derivations implies that their domain is a Banach algebra, and by the symmetry of \mathcal{A} the domain is inverse-closed in \mathcal{A} , see [15].

A more refined concept of smoothness can be developed, if \mathcal{A} is invariant under the bounded action of a d -dimensional automorphism group. In this case algebras of Bessel-Besov type can be defined, and the properties of the group action imply that the spaces defined form inverse-closed subalgebras of \mathcal{A} , see [20].

A different approach to smoothness is by approximation using approximation schemes adapted to the algebra multiplication. This line of research, initiated by Almira and Luther [2, 3], yields Banach algebras of approximation spaces that are inverse-closed in \mathcal{A} , if \mathcal{A} is symmetric [15].

Moreover, if \mathcal{A} is invariant under the action of the translation group and the approximation scheme consists of the *bandlimited elements* of \mathcal{A} , we obtain Jackson-Bernstein theorems that identify approximation spaces of polynomial order with Besov spaces.

All of the above has been carried out in two previous publications [15, 20] for smoothness spaces of finite order. Now we use the same principles to construct inverse-closed subalgebras of ultradifferentiable elements.

Date: January 17, 2012.

1991 Mathematics Subject Classification. 41A65, 42A10, 47B47.

Key words and phrases. Banach algebra, inverse closedness, spectral invariance, Carleman classes, Dales-Davie algebras, matrix algebra, off-diagonal decay, automorphism group.

A. K. was supported by National Research Network S106 SISE of the Austrian Science Foundation (FWF) and the FWF project P22746N13.

Classes of Carleman type are defined by growth conditions on the norms of higher derivations in the same way as for functions, and we obtain a characterization of inverse-closed Carleman classes by adapting a proof of Siddiqi [31]. If the growth of the derivations satisfies the condition (M2') of Komatsu, then an alternative description of the Carleman classes as union of weighted spaces or approximation spaces is available.

Whereas Carleman algebras are inductive limits of Banach spaces we can also define *Banach algebras* of ultradifferentiable elements derived from a given Banach algebra. The construction generalizes an approach used by Dales and Davie [7] for functions defined on perfect subsets of the complex plane, so we call the resulting Banach algebras *Dales-Davie algebras*. An result of Honary and Abtahi [1] on inverse-closed Dales-Davie algebras of functions can be adapted to the noncommutative situation (Theorem 32).

The general theory has applications to Banach algebras of matrices with off-diagonal decay. The formal commutator $\delta(A) = [X, A]$, $X = 2\pi i \text{Diag}((k)_{k \in \mathbb{Z}})$, is a derivation on $\mathcal{B}(\ell^2)$, and its domain defines an algebra of matrices with off-diagonal decay that is inverse-closed in $\mathcal{B}(\ell^2)$ [15, 3.4]. The translation group acts boundedly on $\mathcal{B}(\ell^2)$ by conjugation with the modulation operator $M_t = \text{Diag}(e^{2\pi i k \cdot t})_{k \in \mathbb{Z}^d}$,

$$(1) \quad \chi_t(A) = M_t A M_{-t} = \sum_{k \in \mathbb{Z}^d} \hat{A}(k) e^{2\pi i k \cdot t} \quad \text{for } t \in \mathbb{R}^d,$$

where $\hat{A}(k)$ is the k th side diagonal of A ,

$$(2) \quad \hat{A}(k)(l, m) = \begin{cases} A(l, m), & l - m = k, \\ 0, & \text{otherwise.} \end{cases}$$

In [15, 20] the theory of smooth and inverse-closed subalgebras has been applied to describe Banach algebras of matrices with off-diagonal decay.

The approximation theoretic characterization of Carleman classes of Gevrey type on $\mathcal{B}(\ell^2)$ yields a new proof of a result of Demko, Smith and Moss [10].

Theorem 1. *If $A \in \mathcal{B}(\ell^2)$ with $|A(k, l)| \leq C e^{-\gamma|k-l|}$ for constants $C, \gamma > 0$ and all $k, l \in \mathbb{Z}^d$, and if $A^{-1} \in \mathcal{B}(\ell^2)$, then there exist $C', \gamma' > 0$ such that*

$$|A^{-1}|(k, l) \leq C' e^{-\gamma'|k-l|} \quad \text{for all } k, l \in \mathbb{Z}^d.$$

In some instances, Dales-Davie algebras of matrices can be identified with known Banach algebras of matrices, e.g. if $\mathcal{C}_{v_0}^1$ consists of matrices with norm

$$\|A\|_{\mathcal{C}_{v_0}^1} = \sum_{k \in \mathbb{Z}^d} \sum_{l \in \mathbb{Z}^d} |A(l, l - k)|,$$

then $D_M^1(\mathcal{C}_{v_0}^1)$ is a weighted form of this algebra for a submultiplicative weight v_M associated to M , see Section 4.

The organization of the paper is as follows. First we recall some facts from the theory of Banach algebras and review results of [15, 20] on inverse-closed subalgebras of a given Banach algebra defined by derivations, automorphism groups, and approximation spaces. In Section 3, after treating C^∞ classes, ultradifferentiable classes of Carleman type are introduced, and necessary and sufficient conditions on their inverse-closedness are given. Carleman classes satisfying axiom (M2') of Komatsu are characterized by approximation and weight conditions. As an application we generalize the result of Demko [10] on the inverses of matrices with exponential off-diagonal decay. The results on the inverse-closedness of Dales-Davie algebras are treated in Section 4. In Section 5 some applications to matrix algebras with off-diagonal decay are given. In the appendix a combinatorial Lemma on the iterated quotient rule is proved.

Acknowledgment: The author wants to thank Karlheinz Gröchenig for many helpful discussions.

2. PRELIMINARIES

2.1. Notation. The cardinality of a finite set A is $|A|$. The d -dimensional torus is $\mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d$. The symbol $\lfloor x \rfloor$ denotes the greatest integer smaller or equal to the real number x . Positive constants will be denoted by C, C', C_1, c , etc., where the same symbol might denote different constants in each equation.

We use the standard multi-index notation. Multi-indices are denoted by Greek letters and are a d -tuples of nonnegative integers. The degree of $x^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d}$ is $|\alpha| = \sum_{j=1}^d \alpha_j$, and $D^\alpha f(x) = \partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d} f(x)$ is the partial derivative. The inequality $\beta \leq \alpha$ means that $\beta_j \leq \alpha_j$ for all indices j . The p -norm on \mathbb{C}^d is denoted by $|x|_p = (\sum_{k=1}^d |x(k)|^p)^{1/p}$.

A submultiplicative weight on \mathbb{Z}^d is a positive function $v : \mathbb{Z}^d \rightarrow \mathbb{R}$ such that $v(0) = 1$ and $v(x+y) \leq v(x)v(y)$ for $x, y \in \mathbb{Z}^d$. The standard polynomial weights are $v_r(x) = (1 + |x|)^r$ for $r \geq 0$. The weighted spaces $\ell_w^p(\mathbb{Z}^d)$ are defined by the norm $\|x\|_{\ell_w^p(\mathbb{Z}^d)} = \|xw\|_{\ell^p(\mathbb{Z}^d)}$. If $w = v_r$ we will simply write $\|x\|_{\ell_r^p(\mathbb{Z}^d)}$. A weight w on \mathbb{Z}^d satisfies the *Gelfand, Raikov, Shilov (GRS)-condition* if $\lim_{n \rightarrow \infty} w(nx)^{1/n} = 1$ for all $x \in \mathbb{Z}^d$.

The continuous embedding of the normed space X into the normed space Y is denoted as $X \hookrightarrow Y$. The operator norm of a bounded linear mapping $A : X \rightarrow Y$ is $\|A\|_{X \rightarrow Y}$. In the special case of operators $A : \ell^2(\mathbb{Z}^d) \rightarrow \ell^2(\mathbb{Z}^d)$ we write $\|A\|_{\mathcal{B}(\ell^2(\mathbb{Z}^d))} = \|A\|_{\ell^2(\mathbb{Z}^d) \rightarrow \ell^2(\mathbb{Z}^d)}$ or simply $\|A\|_{\mathcal{B}(\ell^2)}$.

We will consider Banach spaces with equivalent norms as equal.

2.2. Inverse closed Banach algebras. All Banach algebras are assumed to be *unital*. To verify that a Banach space \mathcal{A} with norm $\|\cdot\|_{\mathcal{A}}$ is a Banach algebra it is sufficient to prove that $\|ab\|_{\mathcal{A}} \leq C\|a\|_{\mathcal{A}}\|b\|_{\mathcal{A}}$ for some constant C . A Banach algebra \mathcal{A} is a (Banach) $*$ -algebra if it has an isometric involution $*$, i.e., $\|a^*\|_{\mathcal{A}} = \|a\|_{\mathcal{A}}$ for all $a \in \mathcal{A}$. The Banach $*$ -algebra \mathcal{A} is *symmetric*, if $\sigma_{\mathcal{A}}(a^*a) \subseteq [0, \infty)$ for all $a \in \mathcal{A}$, where $\sigma_{\mathcal{A}}(a)$ denotes the spectrum of $a \in \mathcal{A}$. The spectral radius of $a \in \mathcal{A}$ is $\rho_{\mathcal{A}}(a) = \sup\{|\lambda| : \lambda \in \sigma_{\mathcal{A}}(a)\}$.

Definition 2 (Inverse-closedness). If $\mathcal{A} \subseteq \mathcal{B}$ are Banach algebras with common multiplication and identity, we call \mathcal{A} *inverse-closed* in \mathcal{B} , if

$$(3) \quad a \in \mathcal{A} \text{ and } a^{-1} \in \mathcal{B} \quad \text{implies} \quad a^{-1} \in \mathcal{A}.$$

The relation of inverse-closedness is transitive: If \mathcal{A} is inverse-closed in \mathcal{B} and \mathcal{B} is inverse-closed in \mathcal{C} , then \mathcal{A} is inverse-closed in \mathcal{C} .

2.3. Derivations. A *derivation* δ on a Banach algebra \mathcal{A} with *domain* $\mathcal{D} = \mathcal{D}(\delta) = \mathcal{D}(\delta, \mathcal{A})$ a subspace of \mathcal{A} is a closed linear mapping $\delta : \mathcal{D} \rightarrow \mathcal{A}$ that satisfies the Leibniz rule

$$(4) \quad \delta(ab) = a\delta(b) + \delta(a)b \quad \text{for all } a, b \in \mathcal{D}.$$

If \mathcal{A} is a $*$ -algebra, we assume that the derivation and the domain are symmetric, i.e., $\mathcal{D} = \mathcal{D}^*$ and $\delta(a^*) = \delta(a)^*$ for all $a \in \mathcal{D}$. The domain is normed with the graph norm $\|a\|_{\mathcal{D}} = \|a\|_{\mathcal{A}} + \|\delta(a)\|_{\mathcal{A}}$.

Assume that \mathcal{A} is a symmetric Banach algebra with a symmetric derivation δ . If $1 \in \mathcal{D}(\mathcal{A})$, then the (symmetric) Banach algebra $\mathcal{D}(\mathcal{A})$ is inverse-closed in \mathcal{A} . Moreover, δ satisfies the quotient rule $\delta(a^{-1}) = -a^{-1}\delta(a)a^{-1}$, see [15].

In more generality, let $\{\delta_1, \dots, \delta_d\}$ be a set of commuting derivations on \mathcal{A} . The domain of $\delta_{r_1}\delta_{r_2}\dots\delta_{r_n}$, $1 \leq r_j \leq d$ is defined by induction as $\mathcal{D}(\delta_{r_1}\delta_{r_2}\dots\delta_{r_n}) = \mathcal{D}(\delta_{r_1}, \mathcal{D}(\delta_{r_2}\dots\delta_{r_n}))$. For every multi-index α the operator $\delta^\alpha = \prod_{1 \leq k \leq d} \delta_k^{\alpha_k}$ and its domain $\mathcal{D}(\delta^\alpha)$ are well defined. In analogy to $C^\alpha(\mathbb{R}^d)$ we equip $\mathcal{D}(\delta^\alpha)$ with the norm

$$\|a\|_{\mathcal{D}(\delta^\alpha)} = \sum_{\beta \leq \alpha} \|\delta^\beta(a)\|_{\mathcal{A}}.$$

Since δ_j is assumed to be a closed operator on \mathcal{A} , it follows that δ^α is a closed operator on $\mathcal{D}(\delta^\alpha)$.

If \mathcal{A} is symmetric and $\mathbf{1} \in \mathcal{D}(\delta_k)$, $1 \leq k \leq d$, then $\mathcal{D}(\delta^\alpha)$ is inverse-closed in \mathcal{A} . Furthermore, the Banach algebra $\mathcal{A}^{(k)} = \bigcap_{|\alpha| \leq k} \mathcal{D}(\delta^\alpha)$ and the Fréchet algebra $\mathcal{A}^{(\infty)} = C^\infty(\mathcal{A}) = \bigcap_{k=0}^\infty \mathcal{A}^{(k)}$ are inverse-closed in \mathcal{A} [15, 3.7].

2.4. Automorphism Groups. A (d -parameter) *automorphism group* acting on the Banach algebra \mathcal{A} is a set of Banach algebra automorphisms $\Psi = \{\psi_t\}_{t \in \mathbb{R}^d}$ of \mathcal{A} that satisfy $\psi_s \psi_t = \psi_{s+t}$ for all $s, t \in \mathbb{R}^d$ and are uniformly bounded, i.e. $M_\Psi = \sup_{t \in \mathbb{R}^d} \|\psi_t\|_{\mathcal{A} \rightarrow \mathcal{A}} < \infty$. If \mathcal{A} is a $*$ -algebra we assume that Ψ consists of $*$ -automorphisms.

We call $a \in \mathcal{A}$ continuous and write $a \in C(\mathcal{A})$, if $\lim_{t \rightarrow 0} \psi_t(a) = a$.

For $t \in \mathbb{R}^d \setminus \{0\}$ the *generator* δ_t , defined by $\delta_t(a) = \lim_{h \rightarrow 0} \frac{\psi_{ht}(a) - a}{h}$ is a closed derivation, and the domain $\mathcal{D}(\delta_t, \mathcal{A})$ of δ_t is the set of all $a \in \mathcal{A}$ for which this limit exists. If \mathcal{A} is a $*$ -algebra, then δ_t is symmetric.

The action of Ψ is periodic, if $\psi_t = \psi_{t+e_j}$ for all $t \in \mathbb{R}^d$ and all $1 \leq j \leq d$. If the action of Ψ is periodic, we can define Fourier coefficients of $a \in C(\mathcal{A})$ by

$$\hat{a}(k) = \int_{\mathbb{T}^d} \psi_t(A) e^{-2\pi i k \cdot t} dt$$

With the group action Ψ it is possible to define the classical smoothness spaces, see, e.g. [6]. We need the Besov spaces that are defined, using the difference operators $\Delta_t^k = (\psi_t - \text{id})^k$, by the norm

$$\|a\|_{\Lambda_r^p(\mathcal{A})} = \|a\|_{\mathcal{A}} + \left(\int_{\mathbb{R}^d} (|t|^{-r} \|\Delta_t^k a\|_{\mathcal{A}})^p \frac{dt}{|t|^d} \right)^{1/p}$$

for $1 \leq p \leq \infty$ (standard change for $p = \infty$), $r > 0$ and the integer $k > r$ (every choice of k yields an equivalent norm). Algebra properties of $\Lambda_r^p(\mathcal{A})$ are discussed in [20]. In particular, $\Lambda_r^p(\mathcal{A})$ is inverse-closed in \mathcal{A} for all $1 \leq p \leq \infty$ and all $r > 0$, see [20, 3.8].

In a similar spirit Bessel potential spaces are introduced and it can be shown that they form inverse-closed subalgebras of \mathcal{A} [20].

2.5. Approximation Spaces. An *approximation scheme* on the Banach algebra \mathcal{A} is a family $(X_n)_{n \in \mathbb{N}_0}$ of closed subspaces of \mathcal{A} that satisfy $X_0 = \{0\}$, $X_n \subseteq X_m$ for $n \leq m$, and $X_n \cdot X_m \subseteq X_{n+m}$, $n, m \in \mathbb{N}_0$. If \mathcal{A} is a $*$ -algebra, we assume that $\mathbf{1} \in X_1$ and $X_n = X_n^*$ for all $n \in \mathbb{N}_0$. The n -th *approximation error* of $a \in \mathcal{A}$ by X_n is $E_n(a) = \inf_{x \in X_n} \|a - x\|_{\mathcal{A}}$. For $1 \leq p < \infty$ and w a weight on \mathbb{N}_0 the approximation space $\mathcal{E}_w^p(\mathcal{A})$ consists of all $a \in \mathcal{A}$ for which the norm

$$(5) \quad \|a\|_{\mathcal{E}_w^p} = \left(\sum_{k=0}^\infty E_k(a)^p w(k)^p \right)^{1/p}$$

is finite (standard change for $p = \infty$). If w is a standard polynomial weight, $w = v_r$ for some $r > 0$, then in order to remain consistent with the existing literature we define $\mathcal{E}_r^p(\mathcal{A}) = \mathcal{E}_{v_{r-1/p}}^p(\mathcal{A})$.

Algebra properties of approximation spaces are discussed in [3, 15]. In particular, in [15] the following result is proved.

Proposition 3. *If \mathcal{A} is a symmetric Banach algebra with approximation scheme $(X_n)_{n \in \mathbb{N}_0}$ then $\mathcal{E}_r^p(\mathcal{A})$ is inverse-closed in \mathcal{A} .*

Approximation with bandlimited elements. The relation between smoothness and approximation is given by the Weierstrass theorem and Jackson-Bernstein-theorems.

Given a Banach algebra with automorphism group we say that $a \in \mathcal{A}$ is σ -bandlimited for $\sigma > 0$, if there is a constant C such that for every multi-index α the Bernstein inequality

$$(6) \quad \|\delta^\alpha(a)\|_{\mathcal{A}} \leq C(2\pi\sigma)^{|\alpha|}$$

is satisfied. An element is *bandlimited*, if it is σ -bandlimited for some $\sigma > 0$. In this case $X_0 = \{0\}$, $X_n = \{a \in \mathcal{A} : a \text{ is } n\text{-bandlimited}\}$, $n \in \mathbb{N}$, is an approximation scheme for \mathcal{A} [15, Lemma 5.8].

Theorem 4 (Weierstrass approximation theorem). *If \mathcal{A} is a Banach algebra with automorphism group Ψ , the set of bandlimited elements is dense in $C(\mathcal{A})$.*

Theorem 5 (Jackson-Bernstein-Theorem). *Let \mathcal{A} be a Banach algebra with automorphism group Ψ , and assume that $r > 0$ and $1 \leq p \leq \infty$. If $(X_n)_{n \in \mathbb{N}_0}$ is the approximation scheme of bandlimited elements, then $\Lambda_r^p(\mathcal{A}) = \mathcal{E}_r^p(\mathcal{A})$. In particular*

$$(7) \quad a \in \Lambda_r^\infty(\mathcal{A}) \quad \text{if and only if} \quad E_n(a) \leq Cn^{-r} \text{ for all } n > 0.$$

3. ALGEBRAS OF C^∞ AND ULTRADIFFERENTIABLE ELEMENTS

3.1. C^∞ class. As in the scalar case, elements in a Banach algebra with automorphism group that have derivations of all orders can be characterized by approximation properties.

Proposition 6. *If \mathcal{A} is a Banach algebra with automorphism group Ψ , and $(X_n)_{n \in \mathbb{N}_0}$ is the approximation scheme that consists of the bandlimited elements of \mathcal{A} , then $a \in C^\infty(\mathcal{A})$ if and only if for all $r > 0$ $\lim_{k \rightarrow \infty} E_k(a)k^r = 0$. If the action of Ψ is periodic, this is further equivalent to $\lim_{|k| \rightarrow \infty} \|\hat{a}(k)\|_{\mathcal{A}}|k|^r = 0$ for all $r > 0$.*

Proof. The proof works as for the scalar case. If $a \in C^\infty(\mathcal{A})$, then $a \in \Lambda_{r+1}^\infty(\mathcal{A})$ for any $r > 0$ by the properties of Besov spaces [5, 20]. Using Proposition 5 we conclude that $E_k(a)k^{r+1} \leq C$, and $E_k(a)k^r \rightarrow 0$ for $k \rightarrow \infty$. For the other inclusion observe that (7) implies $a \in \Lambda_r^\infty(\mathcal{A})$, and further $\delta^\alpha a \in \mathcal{A}$ for all α with $|\alpha| = \lfloor r \rfloor$, again by the inclusion relations of Besov spaces [5, 20].

If the action of Ψ is periodic, we use that for all $b \in X_{|k|_\infty}$

$$(8) \quad \hat{a}(k) = \int_{\mathbb{T}^d} (\psi_t(a) - \psi_t(b)) e^{-2\pi i k \cdot t} dt,$$

and so $\|\hat{a}(k)\|_{\mathcal{A}} \leq C\|a - b\|_{\mathcal{A}}$. The infimum of the norm over all $b \in X_{|k|_\infty}$ yields

$$(9) \quad \|\hat{a}(k)\|_{\mathcal{A}} \leq CE_{|k|_\infty}(a),$$

and so $E_k(a)k^r \rightarrow 0$ implies $\|\hat{a}(k)\|_{\mathcal{A}}k^r \rightarrow 0$. If we assume $\|\hat{a}(k)\|_{\mathcal{A}}k^r \rightarrow 0$ for all $r > 0$ then $\sum_{k \in \mathbb{Z}^d} (2\pi i)^k \hat{a}(k)$ converges in the norm of \mathcal{A} to $\delta^\alpha(a)$ for all multi-indices α , as each δ_j is closed in $\mathcal{D}(\delta^\alpha)$, and $a \in C^\infty(\mathcal{A})$. \square

3.2. Carleman Classes.

Definition 7 (cf. [12, 13]). Let \mathcal{A} be a Banach algebra with commuting derivations $\delta_1, \dots, \delta_d$, and let $M = \{M_k\}_{k \in \mathbb{N}_0}$ be a sequence of positive numbers with $M_0 = 1$. For each $r > 0$ we say that $a \in \mathcal{A}$ is in the Banach space $C_{r,M}(\mathcal{A})$, if the norm

$$\|a\|_{C_{r,M}(\mathcal{A})} = \sup_{\alpha \in \mathbb{N}_0^d} \frac{\|\delta^\alpha(a)\|_{\mathcal{A}}}{r^{|\alpha|} M_{|\alpha|}}$$

is finite. The *Carleman Class* $C_M(\mathcal{A})$ is the union of the spaces $C_{r,M}(\mathcal{A})$,

$$C_M(\mathcal{A}) = \bigcup_{r>0} C_{r,M}(\mathcal{A})$$

with the inductive limit topology. Call M the defining sequence of $C_M(\mathcal{A})$.

If $\mathcal{A} = \bigcap_{j=1}^d \ker \delta_j$ we call $C_M(\mathcal{A})$ *trivial*, otherwise $C_M(\mathcal{A})$ is *nontrivial*.

Example 8. If $M_k = 1$ for all k , then $C_{2\pi r, M}(\mathcal{A})$ consists of the r -bandlimited elements of \mathcal{A} . If $M_k = k!^r$ for $r > 0$ then $\mathcal{J}_r(\mathcal{A}) = C_M(\mathcal{A})$ is the *Gevrey-class* of order r . In particular, $\mathcal{J}_1(\mathcal{A})$ consists of the analytic elements of \mathcal{A} , i.e., the elements $a \in \mathcal{A}$ with convergent expansions $\sum_{\alpha \in \mathbb{N}_0^d} \frac{\delta^\alpha(a)}{\alpha!} t^\alpha$ for some $t > 0$. This follows as in the scalar case, see, e.g. [32]. Consequently, if $r \leq 1$ then $\mathcal{J}_r(\mathcal{A})$ consists only of analytic elements.

Equivalence of Defining Sequences. We call two defining sequences M, N *equivalent*, $M \sim N$, if $C_M(\mathcal{A}) = C_N(\mathcal{A})$. If $c^k N_k \leq M_k \leq C^k N_k$ for all indices k and some constants c, C then $M \sim N$. For example, the Gevrey class \mathcal{J}_r is also generated by the sequence $N_k = k^{rk}$.

We recall a standard construction. Let M be a defining sequence. The *function associated to M* is

$$(10) \quad T_M(u) = \sup_{k \geq 0} \frac{u^k}{M_k} \quad \text{for } u > 0.$$

We call T_N and T_M *equivalent* and write $T_N \sim T_M$, if $T_N(cu) \leq T_M(u) \leq T_N(Cu)$ for all $u > 0$ and some positive constants c, C . A function associated to the Gevrey class \mathcal{J}_r is $T_M(u) = \exp(\frac{r}{e} u^{1/r})$.

The *log-convex regularization* M^c of the sequence $M = (M_k)_{k \in \mathbb{N}_0}$ is the largest logarithmically convex sequence smaller than M .

Proposition 9 ([21, 26, 27]). *The log-convex regularization of M satisfies*

$$(11) \quad M_k^c = \sup_{u > 0} \frac{u^k}{T_M(u)}.$$

Moreover, $T_{M^c} = T_M$ and $M^{cc} = M^c$.

We will also need the following simple facts about log-convex sequences.

Lemma 10 ([22, 27]). (1) *For all $k, l \in \mathbb{N}_0$ the sequence M satisfies $M_k^c M_l^c \leq M_{k+l}^c$.* (2) *The sequence $(M_k^c)^{1/k}$ is increasing.*

If $\delta_1, \dots, \delta_d$ are generators of an automorphism group we can give a weak type characterization of $C_{r,M}(\mathcal{A})$.

Lemma 11. *Assume that the automorphism group Ψ acts on \mathcal{A} . An element $a \in \mathcal{A}$ is in $C_{r,M}(\mathcal{A})$ if and only if $G_{a',a}(t) = \langle a', \Psi_t(a) \rangle$, is in $C_{r,M}(L^\infty(\mathbb{R}^d))$ for all $a' \in \mathcal{A}'$, the dual of \mathcal{A} . In this case $\|a\|_{C_{r,M}(\mathcal{A})} \asymp \sup_{\|a'\|_{\mathcal{A}'} \leq 1} \|G_{a',a}\|_{C_{r,M}(L^\infty(\mathbb{R}^d))}$.*

Proof. The required equivalence follows immediately from

$$\|\delta^\alpha a\|_{\mathcal{A}} \leq \sup_{\|a'\|_{\mathcal{A}'} \leq 1} \|G_{a',\delta^\alpha a}\|_{L^\infty(\mathbb{R}^d)} = \sup_{\|a'\|_{\mathcal{A}'} \leq 1} \|D^\alpha G_{a',a}\|_{L^\infty(\mathbb{R}^d)} \leq M_\Psi \|\delta^\alpha a\|_{\mathcal{A}}$$

by dividing with $r^\alpha M_{|\alpha|}$ and taking suprema over all α . The equality $G_{a',\delta^\alpha a} = D^\alpha G_{a',a}$ is a consequence of elementary properties of $G_{a',a}$ [15, Lemma 3.20]. \square

Proposition 12 ([14, 27]). *Assume that the automorphism group Ψ acts on \mathcal{A} , and let M be a defining sequence for $C_M(\mathcal{A})$. If $\liminf_k M_k^{1/k} = 0$, then $C_M(\mathcal{A})$ is trivial. If $0 < \liminf_k M_k^{1/k} < \infty$, then $C_M(\mathcal{A})$ is the class of bandlimited elements. If $\lim_{k \rightarrow \infty} M_k^{1/k} = \infty$, and $(M_k^c)^{1/k} \asymp (N_k^c)^{1/k}$, then $C_M(\mathcal{A}) = C_N(\mathcal{A})$. Moreover, the last condition is equivalent to $T_M \sim T_N$.*

Proof. As $a \in C_M(\mathcal{A})$ if and only if $G_{a',a} \in C_M(\mathbb{R}^d)$ for all $a' \in \mathcal{A}'$, the conditions follow from [27, 6.5.III] by a weak type argument. The statement given there is for functions on the real line, but it remains true for functions on \mathbb{R}^d . In the proof one has to replace the Kolmogorov inequality [26, 6.3.III] by the Cartan-Gorny estimates [27, (6.4.5)]. They can be verified for functions on \mathbb{R}^d as well (see [22, IV.E., Problem 7]).

The equivalence between condition (12) and $T_M \sim T_N$ follows directly from the definition of equivalent associated functions. \square

Corollary 13. *In particular, we obtain that $C_M(\mathcal{A}) = C_{M^c}(\mathcal{A})$.*

Algebra properties of Carleman classes. In this section we verify that $C_M(\mathcal{A})$ is an inverse-closed subalgebra of \mathcal{A} , if $C_M(\mathcal{A}) = C_{M^c}(\mathcal{A})$. If \mathcal{A} has an automorphism group this follows from Proposition 12.

Proposition 14. *Each Carleman class $C_M(\mathcal{A})$ is an algebra.*

Proof. The proof is as in Komatsu [21]. \square

We need the following technical term: A sequence $(u_k)_{k \in \mathbb{N}_0}$ of positive numbers is almost increasing, if $u_k \leq C u_l$ for all $k < l$ and a constant $C > 0$.

Lemma 15. *Assume that the defining sequence M satisfies $M = M^c$. The sequence $(M_k/k!)^{1/k}$ is almost increasing if and only if there is a $C > 0$ such that for all $l \in \mathbb{N}$ and all indices $j_k, k = 1, \dots, l$ with $j = \sum_{k=1}^l j_k$*

$$(12) \quad \prod_{k=1}^l \frac{M_{j_k}}{j_k!} \leq C^j \frac{M_j}{j!}.$$

Proof. Assuming that $(M_k/k!)^{1/k}$ is almost increasing we obtain

$$\frac{M_{j_k}}{j_k!} \leq C^k \left(\frac{M_j}{j!} \right)^{k/j},$$

and the “if” part follows by multiplying these estimates. For the other implication observe first that Stirling’s formula implies that $(M_k/k!)^{1/k}$ is almost increasing if and only if there is a $C' > 0$ such that

$$(13) \quad \frac{M_k^{1/k}}{k} \leq C' \frac{M_l^{1/l}}{l} \quad \text{for all } k < l.$$

If $l = rk$ for an integer r then (12) implies

$$\frac{M_k^{1/k}}{k} \leq C' \frac{M_{rk}^{1/rk}}{rk}.$$

If $rk < l < (r+1)k$, we use an interpolation argument. By Lemma 10 the sequence $M_k^{1/k}$ is increasing in k , so

$$\frac{M_l^{1/l}}{l} \geq \frac{M_{kr}^{1/kr}}{kr} \frac{kr}{l} \geq \frac{kr}{l} \frac{1}{C'} \frac{M_k^{1/k}}{k}$$

by what has been just proved. But this implies

$$\frac{M_k^{1/k}}{k} \leq C' \frac{l}{kr} \frac{M_l^{1/l}}{l} \leq 2C' \frac{M_l^{1/l}}{l}. \quad \square$$

\square

Remark. (a) For the proof of the direct implication we do not need the condition that $M = M^c$. (b) Equation (13) implies that $M_k^{1/k} \rightarrow \infty$ if $(M_k/k!)^{1/k}$ is almost increasing.

Theorem 16 ([25, 31]). *If $C_M(\mathcal{A}) = C_{M^c}(\mathcal{A})$ and if $(M_k/k!)^{1/k}$ is almost increasing, then $C_M(\mathcal{A})$ is inverse-closed in \mathcal{A} .*

We adapt the method of [31] to the noncommutative situation. We need a form of the iterated quotient rule that will be proved in the appendix.

Lemma 17. *Let $E = \{1, \dots, d\}$ and $\delta_1, \dots, \delta_d$ be derivations that satisfy the quotient rule*

$$\delta_j(a^{-1}) = -a^{-1} \delta_j(a) a^{-1} \quad \text{for all } j \in E.$$

For every $k \in \mathbb{N}$ and every tuple $B = (b_1, \dots, b_k) \in E^k$ set $\delta_B(a) = \delta_{b_1} \dots \delta_{b_k}(a)$. Define the ordered partitions of B into m nonempty subtuples as

$$P(B, m) = \{(B_1, \dots, B_m) : B = (B_1, \dots, B_m), B_i \neq \emptyset \text{ for all } i\}.$$

Then

$$(14) \quad \delta_B(a^{-1}) = \sum_{m=1}^{|B|} (-1)^m \sum_{(B_i)_{1 \leq i \leq m} \in P(B, m)} \left(\prod_{j=1}^m a^{-1} \delta_{B_i}(a) \right) a^{-1}.$$

Proof of Theorem 16. Assume that $|\alpha| = k$. With the notation of Lemma 17 there is a k -tuple B with $|B| = k$ such that $\delta^\alpha = \delta_B$. As $a \in C_M(\mathcal{A})$, we know that $\|\delta_{B_i}(a)\| \leq A r^{|B_i|} M_{|B_i|}$ for some constants $C, r > 0$. The number of (nonempty) partitions of B into sets $(B_i)_{1 \leq i \leq m} \in P(B, m)$ of cardinality k_i is $\binom{k}{k_1, \dots, k_m}$, so we obtain the norm estimate

$$(15) \quad \begin{aligned} \|\delta^\alpha(a^{-1})\|_{\mathcal{A}} &\leq \sum_{m=1}^k \|a^{-1}\|_{\mathcal{A}}^{m+1} \sum_{\substack{k_1 + \dots + k_m = k \\ k_j \geq 1}} \binom{k}{k_1, \dots, k_m} \left(\prod_{j=1}^m C r^{k_j} M_{k_j} \right) \\ &= r^k \sum_{m=1}^k \|a^{-1}\|_{\mathcal{A}}^{m+1} C^m \sum_{\substack{k_1 + \dots + k_m = k \\ k_j \geq 1}} \binom{k}{k_1, \dots, k_m} \left(\prod_{j=1}^m M_{k_j} \right) \end{aligned}$$

Using (12) we obtain

$$\begin{aligned} \|\delta^\alpha(a^{-1})\|_{\mathcal{A}} &\leq r^k C^k M_k \sum_{m=1}^k \|a^{-1}\|_{\mathcal{A}}^{m+1} A^m \sum_{\substack{k_1 + \dots + k_m = k \\ k_j \geq 1}} 1 \\ &= r^k C^k M_k \sum_{m=1}^k \|a^{-1}\|_{\mathcal{A}}^{m+1} A^m \binom{k-1}{m-1} \leq C_1^k M_k, \end{aligned}$$

and this is what we wanted to show. \square

Corollary 18. *The Gevrey classes $\mathcal{G}_r(\mathcal{A})$ are inverse-closed in \mathcal{A} , if $r \geq 1$.*

3.3. Description by Weighted and Approximation Spaces. In this section we characterize Carleman classes by unions of weighted spaces and of approximation spaces, if the action of the automorphism group Ψ on the Banach algebra \mathcal{A} is periodic and the sequence M satisfies Komatsu's condition (M2').

Definition 19. Let \mathcal{A} be a Banach $*$ -algebra with periodic automorphism group Ψ . For $1 \leq p \leq \infty$ and a weight v on \mathbb{Z}^d we introduce the weighted spaces

$$\mathcal{C}_v^p(\mathcal{A}) = \{a \in \mathcal{A} : \|a\|_{\mathcal{C}_v^p(\mathcal{A})} = \left(\sum_{k \in \mathbb{Z}^d} \|\hat{a}(k)\|_{\mathcal{A}}^p v(k)^p \right)^{1/p} < \infty\}$$

with the obvious modification for $p = \infty$, where $\hat{a}(k)$ are the Fourier coefficients of a (see Section 2.4).

Remark. If $\ell_v^p(\mathbb{Z}^d)$ is a Banach algebra with respect to convolution, then $\mathcal{C}_v^p(\mathcal{A})$ is an inverse-closed subalgebra of \mathcal{A} . The proof is a straightforward adaption of the proof of [17, Theorem 3.2], based on the theorem of Bochner-Philips.

Lemma 20. *If M is a defining sequence for $C_M(\mathcal{A})$, $r > 0$, and $T_{r,M}(k) = T_M(\frac{2\pi|k|_\infty}{r})$, then $C_{T_{r,M}}^1(\mathcal{A}) \subseteq C_{r,M}(\mathcal{A}) \subseteq C_{T_{r,M}}^\infty(\mathcal{A})$.*

Proof. Assume first that $a \in C_{r,M}(\mathcal{A})$. Let j be an index such that $|k_j| = |k|_\infty$. Then, by l -fold partial integration

$$\hat{a}(k) = \int_{\mathbb{T}^d} \psi_l(a) e^{-2\pi i k \cdot t} dt = \frac{1}{(2\pi i k_j)^l} \int_{\mathbb{T}^d} \psi_l(\delta_{e_j}^l a) e^{-2\pi i k \cdot t} dt.$$

Taking norms we obtain

$$\|\hat{a}(k)\|_{\mathcal{A}} \leq C \frac{r^l M_l}{(2\pi|k|_\infty)^l}.$$

This relation is valid for all $l \in \mathbb{N}_0$, and therefore also for the infimum, which yields $\|\hat{a}(k)\|_{\mathcal{A}} \leq C/T_{r,M}(k)$, or $a \in \mathcal{C}_{T_{r,M}}^\infty(\mathcal{A})$.

For the converse inclusion assume that $a \in \mathcal{C}_{T_{r,M}}^1$, i.e., $\sum_{k \in \mathbb{Z}^d} \|\hat{a}(k)\|_{\mathcal{A}} T_{r,M}(k) < \infty$. For $\alpha \in \mathbb{N}_0^d$ we estimate the norm of $\delta^\alpha(a)$ by

$$\begin{aligned} \|\delta^\alpha(a)\|_{\mathcal{A}} &\leq \sum_{k \in \mathbb{Z}^d} \|\delta^\alpha(\hat{a}(k))\|_{\mathcal{A}} \leq \sum_{k \in \mathbb{Z}^d} (2\pi|k|_\infty)^{|\alpha|} \|\hat{a}(k)\|_{\mathcal{A}} \\ &\leq \|a\|_{\mathcal{C}_{T_{r,M}}^1(\mathcal{A})} \sup_{k \in \mathbb{Z}^d} \frac{(2\pi|k|_\infty)^{|\alpha|}}{T_{r,M}(k)} \leq \|a\|_{\mathcal{C}_{T_{r,M}}^1(\mathcal{A})} \sup_{u>0} \frac{u^{|\alpha|}}{T_M(u/r)} \\ &= \|a\|_{\mathcal{C}_{T_{r,M}}^1(\mathcal{A})} r^{|\alpha|} M_{|\alpha|}^c, \end{aligned}$$

the last equality by (11), and so $a \in C_{r,M^c}(\mathcal{A}) = C_{r,M}(\mathcal{A})$. \square

Corollary 21. *With the notation of Lemma 20,*

$$\bigcup_{r>0} \mathcal{C}_{T_{r,M}}^1(\mathcal{A}) \hookrightarrow C_M(\mathcal{A}) \hookrightarrow \bigcup_{r>0} \mathcal{C}_{T_{r,M}}^\infty(\mathcal{A}),$$

where all spaces are equipped with their natural inductive limit topologies.

In order to obtain equality in Corollary 21 we impose condition (M2') of Komatsu [21].

Lemma 22 ([28], [21, Prop. 3.4]). *If M is a defining sequence, the following are equivalent:*

- (M2') *There exist constants $c > 0$, $h > 1$ such that for all $k \in \mathbb{N}$.*
 - (1) $T_M(hr) \geq CrT_M(r)$ for all $r > 0$.
 - (2) $\frac{T_M(\lambda r)}{T_M(r)} \geq \exp(\log(r/c) \log \lambda / \log h)$ for all $r, \lambda > 0$.

Example 23. The defining sequence for the Gevrey-class \mathcal{J}_r , $r > 0$ satisfies (M2').

Proposition 24. *If \mathcal{A} is a Banach algebra with periodic automorphism group and if the defining sequence satisfies (M2')*

$$C_M(\mathcal{A}) = \bigcup_{r>0} \mathcal{C}_{T_{r,M}}^1(\mathcal{A}) = \bigcup_{r>0} \mathcal{C}_{T_{r,M}}^\infty(\mathcal{A}) = \bigcup_{r>0} \mathcal{E}_{T_{r,M}}^\infty(\mathcal{A})$$

with the interpretation that these algebras are topologically isomorphic.

Proof. (see, e.g., [23]) We split the proof into several parts. By known properties of inductive limits [11] it is sufficient to prove the following inclusions.

- (1) $\mathcal{C}_{T_{r,M}}^1(\mathcal{A}) \hookrightarrow \mathcal{C}_{T_{r,M}}^\infty(\mathcal{A})$: This follows from Lemma 20.
- (2) $\mathcal{C}_{T_{\lambda r,M}}^\infty(\mathcal{A}) \hookrightarrow \mathcal{C}_{T_{r,M}}^1(\mathcal{A})$ for some $\lambda > 0$: Using Lemma 22, (2), we obtain the estimate

$$\begin{aligned} \sum_{k \in \mathbb{Z}^d} \|\hat{a}(k)\|_{\mathcal{A}} T_{r,M}(|k|_\infty) &\leq \sum_{k \in \mathbb{Z}^d} \|\hat{a}(k)\|_{\mathcal{A}} T_{r,M}(\lambda|k|_\infty) \exp\left(-\log\left(\frac{|k|_\infty}{c}\right) \frac{\log \lambda}{\log h}\right) \\ &\leq \sup_{k \in \mathbb{Z}^d} \|\hat{a}(k)\|_{\mathcal{A}} T_{r,M}(\lambda|k|_\infty) \sum_{k \in \mathbb{Z}^d} \left(\frac{|k|_\infty}{c}\right)^{-\log \lambda / \log h}. \end{aligned}$$

If we choose λ such that $\log \lambda / \log h > d$, the sum on the right hand side of the inequality is convergent.

- (3) $\mathcal{E}_{T_{r,M}}^\infty(\mathcal{A}) \hookrightarrow \mathcal{C}_{T_{r,M}}^\infty(\mathcal{A})$ follows from (9).
- (4) $\mathcal{C}_{T_{r,M}}^\infty(\mathcal{A}) \hookrightarrow \mathcal{E}_{T_{\kappa r,M}}^\infty(\mathcal{A})$ for some $\kappa > 0$ will be verified without loss of generality for $r = 2\pi$. The approximation error of $a \in \mathcal{C}_{T_{2\pi,M}}^\infty(\mathcal{A})$ can be estimated by $E_l(a) \leq \sum_{|k|_\infty \geq l} \|\hat{a}(k)\|_{\mathcal{A}} \leq \|a\|_{\mathcal{C}_{T_{2\pi,M}}^\infty} \sum_{|k|_\infty \geq l} T_{2\pi,M}^{-1}(|k|)$. As $T_{2\pi,M}(u) = T_M(u)$ is increasing, we can replace the sum by

an integral. We assume that l is so large that $\frac{\log(l/c)}{\log h} > 2d$, and obtain

$$\begin{aligned} \sum_{|k|_\infty \geq l} T_M^{-1}(|k|) &\leq \int_{|k|_\infty \geq l} T_M^{-1}(|k|) dk \leq C' \int_l^\infty \frac{1}{T_M(u)} u^{d-1} du \\ &= C' l^d \int_1^\infty \frac{1}{T_M(lv)} v^{d-1} dv \leq C' \frac{l^d}{T_M(l)} \int_1^\infty v^{d-1} e^{-\frac{\log(l/c) \log v}{\log h}} dv \\ &= C' \frac{l^d}{T_M(l)} \int_1^\infty v^{d-1-\frac{\log(l/c)}{\log h}} dv = C' \frac{l^d}{T_M(l)} \frac{1}{\frac{\log(l/c)}{\log h} - d} \leq C' \frac{l^d}{T_M(l)} d^{-1}, \end{aligned}$$

where we have used (2) of Lemma 22 in the second line. Applying (1) of Lemma 22 d times we obtain $T_M(l) \geq Cl^dT(l/h^d)$ with a constant C independent of l . Substituting this in the current estimate we obtain $E_l(a) \leq C\|a\|_{\mathcal{C}_{2\pi,M}^\infty} T_M^{-1}(l/h^d)$, and the constant C is independent of l . So $\|a\|_{\mathcal{E}_{2\pi h^d,M}^\infty} \leq C\|a\|_{\mathcal{C}_{2\pi,M}^\infty}$, and that is what we wanted to show. \square

For a more general discussion of approximation results see [29].

4. DALES-DAVIE ALGEBRAS

In this section we assume that Ψ is a *one* parameter automorphism group acting on the Banach algebra \mathcal{A} .

We define Banach algebras that are determined by growth conditions on the sequence $(\|\delta^k(a)\|_{\mathcal{A}})_{k \in \mathbb{N}_0}$ by adapting a similar construction introduced in [7] for scalar functions in the complex plane.

Definition 25. Let $M = (M_k)_{k \geq 0}$ be an *algebra sequence*, that is, a sequence of positive numbers with $M_0 = 1$ and $\frac{M_{k+l}}{(k+l)!} \geq \frac{M_k}{k!} \frac{M_l}{l!}$ for all $k, l \in \mathbb{N}_0$. The *Dales-Davie algebra* $D_M^1(\mathcal{A})$ consists of the elements $a \in \mathcal{A}$ with finite norm

$$\|a\|_{D_M^1(\mathcal{A})} = \sum_{k=0}^{\infty} M_k^{-1} \|\delta^k(a)\|_{\mathcal{A}}.$$

The space $D_M^1(\mathcal{A})$ is indeed a Banach algebra. This will be proved in Proposition 28.

Example 26. Recall that the norm of a derivation on $\mathcal{C}_{v_0}^1(\mathcal{A})$ (see Definition 19) is $\|\delta^k(a)\|_{\mathcal{C}_{v_0}^1(\mathcal{A})} = \sum_{l \in \mathbb{Z}} \|\hat{a}(l)\|_{\mathcal{A}} (2\pi|l|)^k$. For the norm on $D_M^1(\mathcal{C}_{v_0}^1(\mathcal{A}))$ we obtain

$$\|a\|_{D_M^1(\mathcal{A})} = \sum_{k=0}^{\infty} M_k^{-1} \sum_{l \in \mathbb{Z}} \|\hat{a}(l)\|_{\mathcal{A}} (2\pi|l|)^k = \sum_{l \in \mathbb{Z}} \|\hat{a}(l)\|_{\mathcal{A}} \sum_{k=0}^{\infty} \frac{(2\pi|l|)^k}{M_k}.$$

Let us define the weight v_M associated to M by

$$(16) \quad v_M(l) = \sum_{k=0}^{\infty} \frac{(2\pi|l|)^k}{M_k}.$$

Thus we obtain $D_M^1(\mathcal{C}_{v_0}^1(\mathcal{A})) = \mathcal{C}_{v_M}^1(\mathcal{A})$. For this example we have established a relation between the growth of derivatives and weights.

We recall some notions from complex analysis (see, e.g. [24]). For an entire function f let $M_f(r) = \sup_{|x| \leq r} |f(x)|$. The *order* of f is $\rho_f = \lim_{r \rightarrow \infty} \log \log M_f(r) / \log r$. If f has finite order ρ_f , the *type* of f is $\sigma_f = \lim_{r \rightarrow \infty} \log M_f(r) r^{-\rho_f}$. If $\sigma_f = 0$, we say that f has minimal type.

In the following lemma some basic properties of v_M are collected.

Lemma 27.

- (1) If M is an algebra sequence, then $v_M(|k|)$ is submultiplicative.

- (2) The weight v_M can be extended from the positive semiaxis to an entire function if and only if $\lim_{k \rightarrow \infty} M_k^{1/k} = \infty$.
- (3) The weight v_M satisfies the GRS condition if and only if $\lim_{k \rightarrow \infty} (M_k/k!)^{1/k} = \infty$. Furthermore, v_M is GRS if and only if the analytic continuation of v_M is an entire function of order $\rho_{v_M} \leq 1$, and, if $\rho_{v_M} = 1$, then v_M is of minimal type.

Proof. (1) Let $r, s \geq 0$. Then

$$v_M(r+s) = \sum_{k=0}^{\infty} \frac{(2\pi)^k}{M_k} \sum_{l=0}^k \frac{k!}{l!(k-l)!} r^l s^{k-l} \leq \sum_{k=0}^{\infty} \sum_{l=0}^k \frac{(2\pi r)^l}{M_l} \frac{(2\pi s)^{k-l}}{M_{k-l}} \leq v_M(r) v_M(s).$$

As v_M is increasing on \mathbb{R}_0^+ , $v_M(|r+s|) \leq v_M(|r|+|s|)$, and this proves (1) for all values of $r, s \geq 0$.

(2) is a consequence of the Cauchy-Hadamard formula for the convergence radius of the power series v_M .

(3) We use the following formulas for order and type of the entire function $f(x) = \sum_{k=0}^{\infty} a_k x^k$ [24, Theorem 1.2].

$$(17) \quad \rho_f = \overline{\lim}_{k \rightarrow \infty} \frac{k \log k}{\log(1/|a_k|)},$$

$$(18) \quad \sigma_f = \frac{1}{\rho_f e} \overline{\lim}_{k \rightarrow \infty} k |a_k|^{\rho_f/k}.$$

If v_M satisfies the GRS condition then for all $\varepsilon > 0$ there is a $r(\varepsilon)$ such that $1 \leq v_M(r) \leq (1+\varepsilon)^r = \exp(r \log(1+\varepsilon))$ for all $r > r(\varepsilon)$. This implies that $\rho_{v_M} \leq 1$, and if $\rho_{v_M} = 1$, then v_M is of minimal type. To verify the last assertion assume first that $\rho_{v_M} < 1$ and choose $\varepsilon > 0$ so small that $\rho_{v_M} + \varepsilon < 1$. Set $N_k = M_k/(2\pi)^k$. Then (17) implies that $k \log k \leq (\rho_{v_M} + \varepsilon) \log N_k$ for $k > k(\varepsilon)$, and so $k^k < N_k^{\rho_{v_M} + \varepsilon}$. It follows that

$$N_k > (k^k)^{\frac{1}{\rho_{v_M} + \varepsilon}} = k^{k(1+\delta)}$$

for some $\delta > 0$, and therefore

$$(19) \quad k^{-1} N_k^{1/k} > k^{\delta} \rightarrow \infty$$

for $k \rightarrow \infty$. If $\rho_{v_M} = 1$, then v_M is of minimal type, and (18) implies that

$$(20) \quad 0 = \lim_{k \rightarrow \infty} k N_k^{-1/k},$$

and that is what we wanted to show.

If we assume that $\lim_{k \rightarrow \infty} (M_k/k!)^{1/k} = \infty$, then the relations (19) and (20) together with (17) and (18) imply that v_M is of order ≤ 1 . If v_M is of order one, the same relations imply that it is of minimal type. This means that for all $\varepsilon > 0$ there is some $r(\varepsilon)$ such that $v_M(r) \leq (1+\varepsilon)^r$ for all $r > r(\varepsilon)$, so v_M is a GRS weight. \square

Proposition 28. *If \mathcal{A} is a Banach algebra with a one-parameter group of automorphisms acting on \mathcal{A} , and M is an algebra sequence, then $D_M^1(\mathcal{A})$ is a Banach algebra.*

Proof. The algebra property follows from using Lemma 27(1). To prove completeness let a_n be a Cauchy sequence in $D_M^1(\mathcal{A})$. This implies that $\delta^k a_n$ is a Cauchy sequence in \mathcal{A} for all indices k . As δ is a closed operator and \mathcal{A} is complete it follows that there is an $a \in \mathcal{A}$ such that for all $k \geq 0$ the sequence $\delta^k a_n$ converges to $\delta^k a$ in \mathcal{A} . By standard arguments this implies that $a_n \rightarrow a$ in $D_M^1(\mathcal{A})$. \square

Proposition 29. *If \mathcal{A} is a Banach algebra with a one-parameter group of automorphisms Ψ acting on \mathcal{A} , and M is an algebra sequence, then all elements of $D_M^1(\mathcal{A})$ are continuous, $C(D_M^1(\mathcal{A})) = D_M^1(\mathcal{A})$.*

Proof. For $a \in D_M^1(\mathcal{A})$,

$$\|\psi_t(a) - a\|_{D_M^1(\mathcal{A})} \leq \sum_{k=0}^M \frac{\|\psi_t(\delta^k(a)) - \delta^k(a)\|_{\mathcal{A}}}{M_k} + (M_{\Psi} + 1) \sum_{k=M+1}^{\infty} \frac{\|\delta^k(a)\|_{\mathcal{A}}}{M_k}.$$

For $\varepsilon > 0$ given we can choose M such that the second sum in the expansion above is smaller than ε . As $\delta^k(a) \in C(\mathcal{A})$ [15, Proposition.3.15] for all $k \in \mathbb{N}_0$, the first sum can be made small by choosing t small enough. \square

Proposition 30. *The bandlimited elements of \mathcal{A} are dense in $D_M^1(\mathcal{A})$.*

Proof. We have to verify that the bandlimited elements of \mathcal{A} coincide with the bandlimited elements of $D_M^1(\mathcal{A})$. Indeed, if a is bandlimited in \mathcal{A} with bandwidth σ , this means that $\|\delta^k a\|_{\mathcal{A}} \leq C\sigma^k$ for all $k \in \mathbb{N}_0$. This implies that

$$\|\delta^k a\|_{D_M^1(\mathcal{A})} = \sum_{l=0}^{\infty} \frac{\|\delta^{l+k} a\|_{\mathcal{A}}}{M_l} \leq C\sigma^k \sum_{l=0}^{\infty} \frac{\|\delta^l a\|_{\mathcal{A}}}{M_l} = C\sigma^k \|a\|_{D_M^1(\mathcal{A})},$$

so a is bandlimited in $D_M^1(\mathcal{A})$. Assume now that a is bandlimited in $D_M^1(\mathcal{A})$. By definition we obtain $\|\delta^k a\|_{\mathcal{A}} \leq \|\delta^k a\|_{D_M^1(\mathcal{A})} \leq C\sigma^k$, and so a is bandlimited in \mathcal{A} . An application of Weierstrass' theorem (Theorem 4) yields the assertion of the proposition. \square

Proposition 31. $D_M^1(\Lambda_r^1(\mathcal{A})) = \Lambda_r^1(D_M^1(\mathcal{A}))$ for $r > 0$.

Proof. Let $a \in D_M^1(\Lambda_r^1(\mathcal{A}))$, and assume that $l > \lfloor r \rfloor$. Then

$$\begin{aligned} \|a\|_{D_M^1(\Lambda_r^1(\mathcal{A}))} &= \sum_{k=0}^{\infty} M_k^{-1} \|\delta^k a\|_{\Lambda_r^1(\mathcal{A})} = \sum_{k=0}^{\infty} M_k^{-1} \int_0^{\infty} \frac{\|\Delta_t^l \delta^k a\|_{\mathcal{A}}}{|t|^r} \frac{dt}{t} \\ &= \int_0^{\infty} \frac{\|\Delta_t^l a\|_{D_M^1(\mathcal{A})}}{|t|^r} \frac{dt}{t} = \|a\|_{\Lambda_r^1(D_M^1(\mathcal{A}))}, \end{aligned}$$

so $a \in \Lambda_r^1(D_M^1(\mathcal{A}))$. The same calculation shows also the converse inclusion. \square

As $\mathcal{E}_r^1(D_M^1(\mathcal{A})) = \Lambda_r^1(D_M^1(\mathcal{A}))$ by (7), Proposition 31 identifies the approximation spaces $\mathcal{E}_r^1(D_M^1(\mathcal{A}))$ with the Dales-Davie algebras over $\Lambda_r^1(\mathcal{A})$.

As $\mathcal{C}_v^1(\mathcal{A})$ is inverse-closed in \mathcal{A} , if v is a GRS weight it would be natural to conjecture that $D_M^1(\mathcal{A})$ is inverse-closed in \mathcal{A} if and only if v_M is a GRS weight. However we can only prove the following.

Theorem 32. *Let \mathcal{A} be a Banach algebra, and M an algebra sequence. Set $P_k = M_k/k!$. If*

$$(21) \quad A_m = \sup \left\{ \left(P_k^{-1} \prod_{j=1}^m P_{l_j} \right)^{1/m} : l_j \geq 1 \text{ for } 1 \leq j \leq m, \sum_{j=1}^m l_j = k \right\}$$

satisfies $\lim_{m \rightarrow \infty} A_m = 0$, then $D_M^1(\mathcal{A})$ is inverse-closed in \mathcal{A} .

Remark. Before proving the theorem we point out how condition (21) is related to the properties of v_M . If (21) is valid, it follows in particular that $\lim_{k \rightarrow \infty} (M_k/k!)^{1/k} = \infty$ (Choose $l_j = 1$ for all j on the RHS of (21)). By Proposition 27 (3) this means that v_M satisfies the GRS condition. Let us assume now that v_M satisfies the GRS condition, or equivalently $\lim_{k \rightarrow \infty} P_k^{1/k} = \infty$. If the sequence P_k is log-convex, then it satisfies (21) by [1, Cor. 3.6]. Choose now a sequence N_k such that $N_k/k!$ is the log-convex minorant of P_k (clearly, $N_k \leq M_k$), then $N_k/k!$ satisfies (21) and is GRS, and $v_N \geq v_M$. This means that condition (21) does not put stronger growth restrictions on v_M than GRS, but rather imposes some sort of regularity condition.

In this respect it would be interesting to solve the equivalence problem for Dales-Davie algebras: What are the conditions for two algebra sequences M and N such that $D_M^1(\mathcal{A}) = D_N^1(\mathcal{A})$?

Proof. We use the iterated quotient rule (14) to estimate the norm of a^{-1} .

$$\begin{aligned}
\|a^{-1}\|_{D_M^1(\mathcal{A})} &= \sum_{k=0}^{\infty} \frac{\|\delta^k(a^{-1})\|_{\mathcal{A}}}{M_k} \\
&\leq \|a^{-1}\|_{\mathcal{A}} + \sum_{k=0}^{\infty} \frac{k!}{M_k} \sum_{m=1}^k \|a^{-1}\|_{\mathcal{A}}^{m+1} \sum_{\substack{l_1+\dots+l_m=k \\ l_j \geq 1}} \prod_{j=1}^m \frac{\|\delta^{l_j}(a)\|_{\mathcal{A}}}{l_j!} \\
&= \|a^{-1}\|_{\mathcal{A}} + \sum_{k=0}^{\infty} \frac{k!}{M_k} \sum_{m=1}^k \|a^{-1}\|_{\mathcal{A}}^{m+1} \sum_{\substack{l_1+\dots+l_m=k \\ l_j \geq 1}} \prod_{j=1}^m P_{l_j} \frac{\|\delta^{l_j}(a)\|_{\mathcal{A}}}{M_{l_j}!} \\
&= \|a^{-1}\|_{\mathcal{A}} + \sum_{m=1}^k \|a^{-1}\|_{\mathcal{A}}^{m+1} \sum_{k=m}^{\infty} \sum_{\substack{l_1+\dots+l_m=k \\ l_j \geq 1}} \frac{1}{P_k} \prod_{j=1}^m P_{l_j} \frac{\|\delta^{l_j}(a)\|_{\mathcal{A}}}{M_{l_j}!} \\
&\leq \|a^{-1}\|_{\mathcal{A}} + \sum_{m=1}^{\infty} \|a^{-1}\|_{\mathcal{A}}^{m+1} A_m^m (\|a\|_{D_M^1(\mathcal{A})} - \|a\|)^m
\end{aligned}$$

By hypothesis, for any $\varepsilon > 0$ there exists an index m_ε , such that $A_m < \varepsilon$ for all $m > m_\varepsilon$. So if ε is small enough the series converges, and $\|a^{-1}\|_{D_M^1(\mathcal{A})} < \infty$. \square

The condition (21) is not easy to verify. Some sufficient conditions on $P_k = M_k/k!$ are given in [1].

Example 33. The Gevrey sequence $M_k = k!^r$, $r > 1$ is an algebra sequence, and $P_k = k!^{r-1}$ is log-convex. By [1, Cor. 3.6] this implies (21), so $D_M^1(\mathcal{A})$ is inverse-closed in \mathcal{A} .

If the algebra \mathcal{A} is *commutative*, we can do better by adapting a proof of Hulanicki [18].

Proposition 34. *If \mathcal{A} is a commutative, symmetric Banach algebra, Ψ a periodic one-parameter group of automorphisms acting on \mathcal{A} , and M a weight sequence that satisfies $\lim_{k \rightarrow \infty} (M_k/k!)^{1/k} = \infty$ (equivalently, v_M is a GRS weight), then $D_M^1(\mathcal{A})$ is inverse-closed in \mathcal{A} .*

Proof. Assume an $\varepsilon > 0$, and decompose $a \in \mathcal{A}$ into $a = a_\sigma + r$, where $\|r\|_{D_M^1(\mathcal{A})} < \varepsilon$, and a_σ is σ -bandlimited for a $\sigma > 0$ that clearly depends on ε . Bernstein's inequality for bandlimited elements (Equation (6)) implies that

$$(22) \quad \|a_\sigma\|_{D_M^1(\mathcal{A})} = \sum_{k=0}^{\infty} \frac{\|\delta^k(a_\sigma)\|_{\mathcal{A}}}{M_k} \leq \sum_{k=0}^{\infty} \frac{(2\pi\sigma)^k}{M_k} \|a_\sigma\|_{\mathcal{A}} = v_M(\sigma) \|a_\sigma\|_{\mathcal{A}}.$$

This implies that

$$\begin{aligned}
\|a^p\|_{D_M^1(\mathcal{A})} &\leq \sum_{l=0}^p \binom{p}{l} \|a_\sigma^l\|_{D_M^1(\mathcal{A})} \varepsilon^{p-l} \leq C \sum_{l=0}^p \binom{p}{l} v_M(l\sigma) \|a_\sigma\|_{\mathcal{A}}^l \varepsilon^{p-l} \\
&\leq C v_M(p\sigma) \sum_{l=0}^p \binom{p}{l} \|a_\sigma\|_{\mathcal{A}}^l \varepsilon^{p-l} = C v_M(p\sigma) (\|a_\sigma\|_{\mathcal{A}} + \varepsilon)^p \\
&\leq C v_M(p\sigma) (\|a\|_{\mathcal{A}} + 2\varepsilon)^p,
\end{aligned}$$

where we have used that a_σ^l is $l\sigma$ -bandlimited. So $\rho_{D_M^1(\mathcal{A})}(a) = \lim_{p \rightarrow \infty} \|a^p\|_{D_M^1(\mathcal{A})}^{1/p} \leq \|a\|_{\mathcal{A}} + 2\varepsilon$, and consequently $\rho_{D_M^1(\mathcal{A})}(a) = \rho_{\mathcal{A}}(a)$. The Lemma of Hulanicki [18, Prop. 2.5] then shows that $D_M^1(\mathcal{A})$ is inverse-closed in \mathcal{A} . \square

5. APPLICATIONS TO MATRIX ALGEBRAS WITH OFF-DIAGONAL DECAY

5.1. Preliminaries. In this section we apply the theory developed so far to inverse-closed subalgebras of infinite matrices with off-diagonal decay. This is continuation of [15, 20].

A *matrix algebra* \mathcal{A} (over \mathbb{Z}^d) is a Banach algebra of matrices that is continuously embedded in $\mathcal{B}(\ell^2(\mathbb{Z}^d))$. We drop the reference to the index set \mathbb{Z}^d whenever possible.

Our examples are matrix algebras with off-diagonal decay. One way to describe off-diagonal decay is by weights. If \mathcal{A} is a matrix algebra and w a weight on \mathbb{Z}^d , the weighted space \mathcal{A}_w consists of the matrices $A \in \mathcal{A}$ such that the matrix A_w with entries $A_w(k, l) = A(k, l)w(k-l)$ is in \mathcal{A} . The norm on \mathcal{A}_w is $\|A\|_{\mathcal{A}_w} = \|A_w\|_{\mathcal{A}}$.

If \mathcal{A} is a matrix algebra over \mathbb{Z}^d , we define the symmetric and commuting derivations $\delta_j(A)(k, l) = [X_j, A](k, l) = 2\pi i(k_j - l_j)A(k, l)$, where $X_j = \text{Diag}(2\pi i k_j)_{k \in \mathbb{Z}^d}$ and $1 \leq j \leq d$.

We call \mathcal{A} *solid*, if $A \in \mathcal{A}$ and $|B(k, l)| \leq |A(k, l)|$ for all indices k, l implies $B \in \mathcal{A}$ and $\|B\|_{\mathcal{A}} \leq \|A\|_{\mathcal{A}}$. If \mathcal{A} is solid, then $\mathcal{A}^{(m)} = \mathcal{A}_{v_m}$ (see Section 2.3 for the definition of $\mathcal{A}^{(m)}$). In particular, \mathcal{A}_{v_m} is an inverse-closed subalgebra of \mathcal{A} .

With the help of the modulation operator $M_t x(k) = e^{2\pi i k \cdot t} x(k)$, $k \in \mathbb{Z}^d$ we define a bounded and periodic group action χ_t on $\mathcal{B}(\ell^2)$ by

$$\chi_t(A) = M_t A M_{-t}, \quad \chi_t(A)(k, l) = e^{2\pi i(k-l) \cdot t} A(k, l).$$

We call the matrix algebra \mathcal{A} *homogeneous* (c. [8, 9], see also [30, Chapter 9]), if the periodic automorphism group $\chi = \{\chi_t\}_{t \in \mathbb{R}^d}$ is uniformly bounded on \mathcal{A} . In this case the derivations δ_j defined above are the generators of the automorphism group.

If \mathcal{A} is a homogeneous matrix algebra, then an easy computation shows that the k th Fourier coefficient of $A \in \mathcal{A}$ coincides with the k th side diagonal, so the notation is consistent.

Example 35. (a) The algebra $\mathcal{B}(\ell^2)$ itself is a homogeneous matrix algebra. (b) If \mathcal{A} is solid, then clearly \mathcal{A} is homogeneous. (c) In the literature the homogeneous matrix algebras $\mathcal{C}_w^p = \mathcal{C}_w^p(\mathcal{B}(\ell^2))$ are often considered (see Definition 19). More examples can be found in [15, 20].

A second possibility to define matrix algebras with off-diagonal decay is by approximation. If \mathcal{A} is a matrix algebra and $\mathcal{T}_N = \{A \in \mathcal{A} : A = \sum_{|k|_\infty < N} \hat{A}(k)\}$ denotes the matrices in \mathcal{A} with bandwidth smaller than N , then $(\mathcal{T}_N)_{N \geq 0}$ is an approximation scheme for \mathcal{A} . In this case the algebras $\mathcal{E}_r^p(\mathcal{A})$ consist of matrices with some kind of off-diagonal decay that in general cannot be expressed by weights, see [15, 20].

If \mathcal{A} is a homogeneous matrix algebra, then a $A \in \mathcal{A}$ is banded with bandwidth N , if and only if it is N -bandlimited with respect to the group action $\{\chi_t\}$, see [15, 5.7].

5.2. Smooth and ultradifferentiable matrix algebras. If we apply Proposition 6 to homogeneous matrix algebras we obtain the following result.

Corollary 36. *If \mathcal{A} is a homogeneous matrix algebra and $A \in \mathcal{A}$, then $\|\hat{A}(k)\|_{\mathcal{A}} = \mathcal{O}(|k|^{-r})$ for all $r > 0$ if and only if $E_t(A) = \mathcal{O}(t^{-r})$ for all $r > 0$. If $A^{-1} \in \mathcal{A}$ then these conditions imply that $\|\widehat{A^{-1}}(k)\|_{\mathcal{A}} = \mathcal{O}(|k|^{-r})$ for all $r > 0$.*

Considering Carleman classes for a homogeneous matrix algebra we want to identify the trivial classes first. If \mathcal{A} is a homogeneous matrix algebra, then $C_M(\mathcal{A})$ is trivial if and only if \mathcal{A} consists only of diagonal matrices.

It is of some interest, that for a special class of homogeneous matrix algebras we obtain a converse of Theorem 16 by adapting a construction in [31, Thm 1].

Proposition 37. *Assume that \mathcal{A} is a homogeneous matrix algebra and that the translation operators T_k defined by $(T_k x)(l) = x(l - k)$ are uniformly bounded:*

$$(23) \quad \|T_k\|_{\mathcal{A}} \leq C$$

for all $k \in \mathbb{Z}^d$. If the nontrivial algebra $C_M(\mathcal{A})$ is inverse-closed in \mathcal{A} , then the defining sequence M is almost increasing.

Remark. (a) Actually, it follows from the proof that it suffices to assume that $\|T_{ke_1}\|_{\mathcal{A}} \leq C$ for all $k \in \mathbb{N}_0$. (b) The condition (23) is equivalent to $\mathcal{C}_{v_0}^1 \hookrightarrow \mathcal{A}$, where $\mathcal{C}_{v_0}^1$ denotes the (unweighted) Baskakov algebra, see Section 5.

Proof. Without loss of generality we may assume that $M = (M_k)_{k \in \mathbb{N}_0}$ is log-convex. By Proposition 12, the condition $\underline{\lim} M_k = 0$ implies that $C_M(\mathcal{A})$ is trivial. If $\underline{\lim} M_k < \infty$ the algebra $C_M(\mathcal{A})$ consists of the banded matrices, which are not inverse-closed in \mathcal{A} . So we may assume that $\lim_{k \rightarrow \infty} M_k^{1/k} = \infty$. Using the convexity polygon with vertices $((k, \log k))_{k \geq 0}$ we can find positive integers $(u_j)_{j \in \mathbb{N}_0}$ with the property that $T_M(u_j) = M_j^{-1} u_j^j$ ([27], see also [22] for a detailed discussion). The matrix $A = \sum_{j=1}^{\infty} 2^{-j} T_M^{-1}(u_j) T_{u_j e_1}$ satisfies

$$\|\delta_1^m(A)\|_{\mathcal{A}} \leq C \sum_{j=1}^{\infty} \frac{(2\pi)^m u_j^m}{2^j T_M(u_j)} \leq C' (2\pi)^m M_m$$

for a constant $C' > 0$. If we choose $\lambda > 1 + \|A\|_{\mathcal{A}}$ then $(\lambda - A)$ is invertible in \mathcal{A} , and, by hypotheses, $(\lambda - A)^{-1} \in C_M(\mathcal{A})$. This means that $\|\delta_1^m(\lambda - A)^{-1}\|_{\mathcal{A}} < Cq^m M_m$ for all $m \in \mathbb{N}_0$ and for constants $C, q > 0$. As \mathcal{A} is a matrix algebra it follows that $\|\delta_1^m(\lambda - A)^{-1}\|_{\mathcal{B}(\ell^2)} < Cq^m M_m$. As A is constant along the diagonals standard facts on convolution operators imply that $\|\delta_1^m(\lambda - A)^{-1}\|_{\mathcal{B}(\ell^2)} = \|D_1^m(\lambda - f)^{-1}\|_{L^\infty(\mathbb{T}^d)}$, where

$$f(t) = \sum_{j=1}^{\infty} 2^{-j} T_M^{-1}(u_j) e^{2\pi i u_j \cdot t_1}.$$

As in [31, Thm 1] we can conclude that $D_1^m f(0) = i^m s_m$, where $s_m > K^{-m} M_m$ for a $K > 1$. Now let us consider the expansion, given by the iterated quotient rule

$$\begin{aligned} D_1^m(\lambda - f)^{-1}(0) &= \sum_{l=1}^m (\lambda - f)^{-l-1}(0) \sum_{\substack{k_1 + \dots + k_l = m \\ k_j \geq 1}} \binom{k}{k_1, \dots, k_l} \prod_{j=1}^l D_1^{k_j} f(0) \\ &= i^m K^{-m} \sum_{l=1}^m (\lambda - f)^{-l-1}(0) \sum_{\substack{k_1 + \dots + k_l = m \\ k_j \geq 1}} \binom{k}{k_1, \dots, k_l} \prod_{j=1}^l s_{k_j} \end{aligned}$$

As $\lambda - f > 0$, we conclude that

$$\sum_{l=1}^m (\lambda - f)^{-l-1}(0) \sum_{\substack{k_1 + \dots + k_l = m \\ k_j \geq 1}} \binom{k}{k_1, \dots, k_l} \prod_{j=1}^l M_{k_j} \leq Cq^m M_m,$$

and, in particular

$$\binom{k}{k_1, \dots, k_l} \prod_{j=1}^l M_{k_j} \leq Cq^m M_m,$$

for $k_1 + \dots + k_l = m$ (actually, we have assumed that $k_j \geq 1$ for all indices j , but the reader can easily verify the last relation for $k_j = 0$ as well). By Lemma 15 this means that M is almost increasing. \square

Corollary 38. *Assume that \mathcal{A} is a homogeneous matrix algebra with uniformly bounded translations. Any nontrivial Carleman class $C_M(\mathcal{A})$ is inverse-closed in \mathcal{A} , if and only if $(M_k/k!)^{1/k}$ is almost increasing.*

The characterization of Carleman classes by weighted spaces and by approximation spaces in Proposition 24 yields a new proof of a result of Demko, Smith and Moss [10] and of Jaffard [19].

Corollary 39. Assume that $A \in \mathcal{B}(\ell^2)$ has off-diagonal decay of exponential order: $|A(k, l)| \leq Ce^{-\gamma|k-l|^r}$ for constants $C, \gamma > 0$, $0 < r \leq 1$, and all $k, l \in \mathbb{Z}^d$. If $A^{-1} \in \mathcal{B}(\ell^2)$, then there exist $C', \gamma' > 0$ such that

$$|A^{-1}(k, l)| \leq C'e^{-\gamma'|k-l|^r} \quad \text{for all } k, l \in \mathbb{Z}^d.$$

Remark. If $r < 1$ better results exist [4, 16].

Dales-Davie algebras of matrices describe new forms of off-diagonal decay. It has already been stated that $\mathcal{C}_{v_M}^1 = D_M^1(\mathcal{C}_{v_0}^1)$. Now consider the following simple corollary of Theorem 32.

Corollary 40. If $A \in \mathcal{B}(\ell^2(\mathbb{Z}))$ satisfies

$$\sum_{k=0}^{\infty} (k!)^{-r} \|\delta^k A\|_{\mathcal{B}(\ell^2)(\mathbb{Z})} < \infty$$

for a $r > 1$, and A has an inverse $A^{-1} \in \mathcal{B}(\ell^2(\mathbb{Z}))$, then this inverse satisfies the same estimate.

The result remains valid, if we replace $(k!)^{-r}$ by any algebra sequence M , such that $M_k/k!$ is log-convex and $\lim_{k \rightarrow \infty} (M_k/k!)^{1/k} = \infty$.

AN ITERATED QUOTIENT RULE

We provide a proof of Lemma 17. The proof is by induction over $|B|$. If $|B| = 1$ there is nothing to prove. Assume that the statement is true for $|B| < k$, and assume $|B| = k$. The Leibniz rule for $\delta_B(a^{-1}a)$ yields

$$(24) \quad \delta_B(a^{-1}a) = 0 = \sum_{(B_1, B_2) \in P(B, 2)} \delta_{B_1}(a^{-1})\delta_{B_2}(a) + a^{-1}\delta_B(a) + \delta_B(a^{-1})a,$$

So

$$(25) \quad \delta_B(a^{-1}) = -a^{-1}\delta_B(a)a^{-1} - \sum_{(B_1, B_2) \in P(B, 2)} \delta_{B_1}(a^{-1})\delta_{B_2}(a)a^{-1}.$$

As $|B_1| < k$ we can apply the induction hypothesis.

$$\begin{aligned} \delta_B(a^{-1}) &= -a^{-1}\delta_B(a)a^{-1} \\ &\quad - \sum_{(B_1, B_2) \in P(B, 2)} \sum_{m=1}^{|B_1|} (-1)^m \sum_{(D_i)_{1 \leq i \leq m} \in P(B_1, m)} \left(\prod_{j=1}^m a^{-1}\delta_{D_j}(a) \right) a^{-1}\delta_{B_2}(a)a^{-1}. \end{aligned}$$

Interchanging the first two summations we obtain

$$\begin{aligned} \delta_B(a^{-1}) &= -a^{-1}\delta_B(a)a^{-1} \\ &\quad - \sum_{m=1}^{k-1} (-1)^m \sum_{\substack{(B_1, B_2) \in P(B, 2) \\ |B_1| \geq m}} \sum_{(D_i)_{1 \leq i \leq m} \in P(B_1, m)} \left(\prod_{j=1}^m a^{-1}\delta_{D_j}(a) \right) a^{-1}\delta_{B_2}(a)a^{-1}. \end{aligned}$$

Now observe that in this expression (D_1, \dots, D_m, B_2) varies over all partitions of B , The condition $(D_i)_{1 \leq i \leq m} \in P(B_1, m)$ already implies that $|B_1| \geq m$, so we can set $D_{m+1} = B_2$ and obtain

$$\begin{aligned} \delta_B(a^{-1}) &= -a^{-1}\delta_B(a)a^{-1} \\ &\quad - \sum_{m=1}^{k-1} (-1)^m \sum_{(D_i)_{1 \leq i \leq m+1} \in P(B, m+1)} \left(\prod_{j=1}^{m+1} a^{-1}\delta_{D_j}(a) \right) a^{-1}. \end{aligned}$$

We change the summation index.

$$\begin{aligned}\delta_B(a^{-1}) &= -a^{-1}\delta_B(a)a^{-1} \\ &\quad + \sum_{l=2}^k (-1)^m \sum_{(D_i)_{1 \leq i \leq l} \in P(B,l)} \left(\prod_{j=1}^l a^{-1}\delta_{D_i}(a) \right) a^{-1}.\end{aligned}$$

The term $-a^{-1}\delta_B(a)a^{-1}$ can be included into the sum for $l = 1$. We obtain

$$\delta_B(a^{-1}) = \sum_{l=1}^k (-1)^m \sum_{(D_i)_{1 \leq i \leq l} \in P(B,l)} \left(\prod_{j=1}^l a^{-1}\delta_{D_i}(a) \right) a^{-1},$$

and this is (14).

REFERENCES

- [1] M. Abtahi and T. G. Honary. On the maximal ideal space of Dales-Davie algebras of infinitely differentiable functions. *Bull. Lond. Math. Soc.*, 39(6):940–948, 2007.
- [2] J. M. Almira and U. Luther. Approximation algebras and applications. In *Trends in approximation theory (Nashville, TN, 2000)*, Innov. Appl. Math., pages 1–10. Vanderbilt Univ. Press, Nashville, TN, 2001.
- [3] J. M. Almira and U. Luther. Inverse closedness of approximation algebras. *J. Math. Anal. Appl.*, 314(1):30–44, 2006.
- [4] A. G. Baskakov. Asymptotic estimates for elements of matrices of inverse operators, and harmonic analysis. *Sibirsk. Mat. Zh.*, 38(1):14–28, i, 1997.
- [5] P. L. Butzer and H. Berens. *Semi-groups of operators and approximation*. Die Grundlehren der mathematischen Wissenschaften, Band 145. Springer-Verlag New York Inc., New York, 1967.
- [6] P. L. Butzer and K. Scherer. *Approximationsprozesse und Interpolationsmethoden*. B. I. Hochschulschriften 826/826a. Bibliographisches Institut, Mannheim, 1968.
- [7] H. G. Dales and A. M. Davie. Quasianalytic Banach function algebras. *J. Functional Analysis*, 13:28–50, 1973.
- [8] K. DeLeeuw. An harmonic analysis for operators. I: Formal properties. *Ill. J. Math.*, 19:593–606, 1975.
- [9] K. DeLeeuw. An harmonic analysis for operators. II: Operators on Hilbert space and analytic operators. *Ill. J. Math.*, 21:164–175, 1977.
- [10] S. Demko, W. F. Moss, and P. W. Smith. Decay rates for inverses of band matrices. *Math. Comp.*, 43(168):491–499, 1984.
- [11] K. Floret and J. Wloka. *Einführung in die Theorie der lokalkonvexen Räume*. Lecture Notes in Mathematics, No. 56. Springer-Verlag, Berlin, 1968.
- [12] V. I. Gorbachuk and M. L. Gorbachuk. *Boundary value problems for operator differential equations*, volume 48 of *Mathematics and its Applications (Soviet Series)*. Kluwer Academic Publishers Group, Dordrecht, 1991. Translated and revised from the 1984 Russian original.
- [13] V. I. Gorbachuk and A. V. Knyazyuk. Boundary values of solutions of operator-differential equations. *Uspekhi Mat. Nauk*, 44(3(267)):55–91, 208, 1989.
- [14] A. Gorny. Contribution à l’étude des fonctions dérivables d’une variable réelle. *Acta Math.*, 71:317–358, 1939.
- [15] K. Gröchenig and A. Klotz. Noncommutative approximation: Inverse-closed subalgebras and off-diagonal decay of matrices. *Constructive Approximation*, 32:429–446, 2010.
- [16] K. Gröchenig and M. Leinert. Symmetry and inverse-closedness of matrix algebras and functional calculus for infinite matrices. *Trans. Amer. Math. Soc.*, 358(6):2695–2711 (electronic), 2006.
- [17] K. Gröchenig and Z. Rzeszutnik. Banach algebras of pseudodifferential operators and their almost diagonalization. *Ann. Inst. Fourier (Grenoble)*, 58(7):2279–2314, 2008.
- [18] A. Hulanicki. On the spectrum of convolution operators on groups with polynomial growth. *Invent. Math.*, 17:135–142, 1972.
- [19] S. Jaffard. Propriétés des matrices “bien localisées” près de leur diagonale et quelques applications. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 7(5):461–476, 1990.
- [20] A. Klotz. Spectral Invariance of Besov-Bessel Subalgebras. *Journal of Approximation Theory*, 164: 268–269, 2012.
- [21] H. Komatsu. Ultradistributions. I. Structure theorems and a characterization. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 20:25–105, 1973.
- [22] P. Koosis. *The logarithmic integral. I*, volume 12 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1988.
- [23] M. Langenbruch. Ultradifferentiable functions on compact intervals. *Math. Nachr.*, 140:109–126, 1989.

- [24] B. Y. Levin. *Lectures on entire functions*, volume 150 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 1996. In collaboration with and with a preface by Yu. Lyubarskii, M. Sodin and V. Tkachenko, Translated from the Russian manuscript by Tkachenko.
- [25] P. Malliavin. Calcul symbolique et sous-algèbres de $L_1(G)$. I, II. *Bull. Soc. Math. France*, 87:181–186, 187–190, 1959.
- [26] S. Mandelbrojt. Analytic functions and classes of infinitely differentiable functions. *Rice Inst. Pamphlet*, 29(1):142, 1942.
- [27] S. Mandelbrojt. *Séries adhérentes, régularisation des suites, applications*. Gauthier-Villars, Paris, 1952.
- [28] H.-J. Petzsche. Die Nuklearität der Ultradistributionsräume und der Satz vom Kern. I. *Manuscripta Math.*, 24(2):133–171, 1978.
- [29] H.-J. Petzsche. Approximation of ultradifferentiable functions by polynomials and entire functions. *Manuscripta Math.*, 48(1-3):227–250, 1984.
- [30] H. S. Shapiro. *Topics in approximation theory*. Springer-Verlag, Berlin, 1971. With appendices by Jan Boman and Torbjörn Hedberg, Lecture Notes in Math., Vol. 187.
- [31] J. A. Siddiqi. Inverse-closed Carleman algebras of infinitely differentiable functions. *Proc. Amer. Math. Soc.*, 109(2):357–367, 1990.
- [32] A. F. Timan. *Theory of approximation of functions of a real variable*. Translated from the Russian by J. Berry. English translation edited and editorial preface by J. Cossar. International Series of Monographs in Pure and Applied Mathematics, Vol. 34. A Pergamon Press Book. The Macmillan Co., New York, 1963.

FACULTY OF MATHEMATICS, UNIVERSITY OF VIENNA, NORDBERGSTRASSE 15, A-1090 VIENNA, AUSTRIA

E-mail address: andreas.klotz@univie.ac.at